Intuitionistic Linear Logic and its Categorical Interpretation

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1 Introduction

Linear logic was proposed by Jean-Yves Girard in 1987 as a "continuation of the constructivization that began with intuitionistic logic"[6]. By carefully restricting structural rules of sequent calculus, Girard was able to achieve a powerful framework for formalizing resource-sensitive reasoning. Linear logic found applications across computer science, natural language processing, linguistics, and proof verification [14].

The core idea of linear logic is to treat logical formulas in a proof not as a collection of truths but as resources that must be consumed exactly once. From here, the scope of interpretation varies: the logic has classical and intuitionistic variants as well as multiplicative, additive, and exponential fragments that can be considered separately. To make the scope of this explainer manageable, we restrict our attention to intuitionistic linear logic (ILL) with finite products.

In this explainer we draw heavily from Melliès' 2009 survey of Categorical Semantics of Linear Logic[9]. This source has established itself as a trusty introductory resource which, appropriately, favors ILL. We also draw heavily from Bierman's dissertation "On Intuitionistic Linear Logic" [2]. The reader is encouraged to consult Melliès' survey and Bierman's dissertation as natural companion pieces to this exposition, as they provides additional context and deeper technical details.

This document is structured as follows: we begin with an overview of sequent calculus followed by a definition of Category of Formulas and Proofs. We then demonstrate that the category is a symmetric monoidal category with finite products, and we also describe how ILL rules define a comonad on the category. Finally, we use all the previously defined pieces to present an complete model of ILL with finite products - something called a Linear Category.

2 What Makes Linear Logic Linear?

2.1 Sequent Calculus Basics

Logic is a formal study of propositions and the proofs that establish their validity. To reason about proofs, we require a flexible framework to express the structure of logical arguments. The tool of choice for many logicians and theoretical computer scientists is *sequent calculus*, as formalized by Gerhard Gentzen. We will consider its intuitionistic variant of propositional logic without quantifiers. In this setting, every proposition, also known as formula, is generated by the following grammar:

$$A, B ::= \top \mid \bot \mid p \mid A \wedge B \mid A \vee B \mid A \Rightarrow B \qquad \text{(SeqCalc)}$$

An intuitionistic *sequent* is written as $\Gamma \vdash A$, where Γ is a sequence of formulas on the left-hand side and A is a single formula on the right-hand side. The sequent $\Gamma \vdash A$ is understood as asserting that "the conjunction of all formulas in Γ implies A," or equivalently, "given assumptions Γ, we can derive exactly one conclusion A." A *formal proof* is then defined as a derivation tree constructed

according to the inference rules of sequent calculus. The leaves of the tree must be axioms and the root represents the sequent to be proven.

2.2 Structural Rules

According to Gentzen's analysis, the rules of a logic can be roughly separated into three classes: *axiom rules*, *structural rules*, and *logical rules*[9]. The axiom rules provide the base cases for derivations, typically including the identity axiom $A \vdash A$. The logical rules govern the introduction and elimination of logical connectives on either side of the sequent. We will return to these in the following section.

Most interesting to us now are structural rules. They operate on the sequent level, regardless of the specific logical structure of the formulas involved. The fundamental structural rules are:

$$\text{Weakening } \frac{\Gamma \vdash B}{\Gamma, A \vdash B} \qquad \text{Contraction } \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \qquad \text{Exchange } \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$$

Once these structural rules were explicitly identified by Gentzen, it became natural to explore what happens when any of them are restricted or removed entirely. These experiments resulted in a class of logics that is called *substructural*, and linear logic is precisely the result of removing the Weakening and Contraction rules from general use.

So why does the absence of these two rules enable the "resource" view on formulas? If we want to assert that every formula will get consumed, like a resource, exactly once, we need to rule out two types of bad behaviors:

1. You cannot create (or destroy) a resource simply because it is not used; every resource must be accounted for. This corresponds to removing Weakening.
2. You cannot consume the same resource more than once; every resource gets destroyed after being consumed. This corresponds to removing Contraction.

While this restriction is nice for modeling resrources, it also comes with a downside: we lose a lot of expressive power. The loss of the two fundamental structural rules means that we will never be able to operate on the level of intuitionistic or classical logic. So, instead of outright removing the rules, linear logic introduces something called an *exponential modality*.

The exponential modality allows us to escape the normal structural rules of linear logic and apply Weakening and Contraction rules at will, but in a very controlled way. This was one of Girard's innovations[6] that made linear logic a truly interesting field of study.

2.2.1 Cut Rule and Its Elimination

There is another famous structural rule called the Cut rule, which allows us to eliminate intermediate lemmas necessary to prove some sequent:

$$\text{Cut } \frac{\Gamma_1 \vdash A \qquad \Gamma_2, A \vdash B}{\Gamma_1, \Gamma_2 \vdash B}$$

What makes this rule interesting is that it is considered *admissible but not necessary* in sequent calculus. In fact, one of the reasons Gentzen's work achieved such prominence was that it came

equipped with a powerful theorem about the cut rule:

Theorem 2.1 (Cut-Elimination Theorem)**.** For any provable sequent $\Gamma \vdash A$ in sequent calculus, there exists a cut-free proof of the same sequent.[15]

Proving cut-elimination is a demanding task: Melliès devotes the majority of Chapter 3 of his survey to present the cut-elimination procedure and establish the theorem for intuitionistic linear logic [9]. This would distract us too much from categorical interpretation of ILL, so we ask the reader to accept this theorem as is.

There is a lot to be said about the relationship between cut-elimination and strong normalization and confluence of proofs. Jumping ahead, the cut-rule corresponds to composition, and therefore eliminating the rule implies the existence of some cut-free, normal form. Bierman's "On Intuitionistic Linear Logic" describes a term assignment system for ILL formulations and demonstrates confluence[2]. This result forms the basis of a number of claims we will make about equivalence of denotations of proofs in the next section.

2.3 Sequent Calculus of Intuitionistic Linear Logic

Now we are ready to present the intuitionistic linear logic with finite products as understood by sequent calculus. We will treat this as ground truth for how ILL *should* behave and categorify these notions in the next sections.

ILL formulas are generated by the following grammar:

$$A, B ::= a \mid 1 \mid \top \mid A \otimes B \mid A \multimap B \mid A \& B \mid !A \tag{ILL}$$

Before diving into the formality, let us develop some resource-oriented intuition for these connectives and exponential modality !:

- The **tensor product** $A \otimes B$ represents having both resources A and B available simultaneously. When we use $A \otimes B$, we must consume both A and B together. Its unit is 1, which represents the "empty resource."
- The **linear implication** $A \multimap B$ represents a process that consumes exactly one copy of A to produce exactly one copy of B. Alternatively, think of it as "converting resource A into B."
- The **with connective** $A \& B$ represents a choice: we have a resource that can be used as either A or B. Whichever does not get picked is to be discarded. Its unit is $\top$, which represents a trivial, always-available resource that requires no input.
- The **exponential modality** $!A$ (pronounced "of course A" or "bang A") represents a resource that can be used any number of times, including zero times or infinitely many times. This modality offers controlled reintroduction of the Weakening and Contraction rules, which is necessary for recovering the full expressiveness of intuitionistic logic.

The complete set of rules for sequent calculus of intuitionistic linear logic with finite products is shown in Figure 1. As we categorify ILL, we will address every rule below and show how it corresponds to morphisms and coherence conditions in the Category of Formulas and Proofs.

$$\text{Axiom}\ \frac{}{A \vdash A} \qquad \text{Cut}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2, A \vdash B}{\Gamma_1, \Gamma_2 \vdash B} \qquad \text{Exchange}\ \frac{\Gamma, A, B \vdash C}{\Gamma, B, A \vdash C}$$

$$\text{Left } 1\ \frac{\Gamma \vdash A}{\Gamma, 1 \vdash A} \qquad \text{Right } 1\ \frac{}{\vdash 1}$$

$$\text{Left } \otimes\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \qquad \text{Right } \otimes\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}$$

$$\text{Left } \multimap\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \multimap B \vdash C} \qquad \text{Right } \multimap\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$$

$$\text{True}\ \frac{}{\Gamma \vdash \top} \qquad \text{Right } \&\ \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \& B}$$

$$\text{Left } \&_1\ \frac{\Gamma, A \vdash C}{\Gamma, A \& B \vdash C} \qquad \text{Left } \&_2\ \frac{\Gamma, B \vdash C}{\Gamma, A \& B \vdash C}$$

$$\text{Promotion}\ \frac{!\Gamma \vdash A}{!\Gamma \vdash !A} \qquad \text{Dereliction}\ \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}$$

$$\text{Weakening}\ \frac{\Gamma \vdash B}{\Gamma, !A \vdash B} \qquad \text{Contraction}\ \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$$

Figure 1: Sequent Calculus of ILL with finite products

3 Categorifying Formulas and Proofs

With sufficient background in place, we are ready to interpret intuitionistic linear logic as a category. This turns out to be quite tricky, particularly when we introduce the exponential modality (denoted by !) in later sections. Luckily, we can make substantial progress by constructing what we call the *Category of Formulas and Proofs*, which will capture most of the structure of ILL. In this section we show how to interpret all connectives except for the exponential modality.

Recall that when we interpreted Simply Typed Lambda Calculus (STλC) in a category with finite products, we showed how each type and typing rule corresponded to a some categorical structure. We will follow a similar path here, except our understanding of the properties of the target category will be enriched as we encounter new connectives.

3.1 Denotations of Formulas and Proofs

Since we are giving semantics to a logic, it only makes sense to categorify it in terms of formulas and proofs, but cannot do so directly. As Melliès points out, to interpret linear logic we need to consider *invariants* of proofs under cut-elimination rather than the proofs themselves.

He defines an invariant to be a map from a specific proof π to it's *denotation* $[\![\pi]\!]$, and the denotation is understood to be a normal form of an ILL proof. As mentioned before, by the Cut-Elimination theorem, we know that every proof containing applications of the Cut rule will have a cut-free normal form.

From here Melliès established several fundamental properties about how the denotations of proofs (and formulas) behave under cut-elimination. These properties will be useful to prove equalities of morphisms.

Theorem 3.1 (Properties of Proof Denotations Under Cut-Elimination)**.**

1 **(Invariance)** The denotation $[\![\pi]\!]$ coincides with the denotation $[\![\pi']\!]$ of any proof π' obtained by applying the cut-elimination procedure to the proof π.

2 **(Modularity)** If a proof π is constructed by applying the Cut rule to subproofs π_1 and π_2, then the denotation of π can be deduced directly from the denotations $[\![\pi_1]\!]$ and $[\![\pi_2]\!]$ of the subproofs. We will see it as $[\![\pi]\!] = [\![\pi_2]\!] \circ [\![\pi_1]\!]$ for composition of morphisms.

3 **(Tensoriality)** If a formula $A \otimes B$ is constructed as a tensor product of A and B, then the denotation of $[\![A \otimes B]\!]$ can be deduced as a tensor product of denotations of subformulas: $[\![A]\!] \otimes [\![B]\!]$. Further, given two proofs

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{A_1 \vdash A_2} \quad \text{and} \quad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{B_1 \vdash B_2}, \text{ the denotation of } \quad \begin{array}{l}\text{Right } \otimes\\ \text{Left } \otimes\end{array} \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{A_1 \vdash A_2} \quad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{B_1 \vdash B_2}}{A_1, B_1 \vdash A_2 \otimes B_2}}{A_1 \otimes B_1 \vdash A_2 \otimes B_2}$$

should follow from the denotations of the proofs π_1 and π_2 as $[\![\pi]\!] = [\![\pi_1]\!] \otimes [\![\pi_2]\!]$. This property is explored in more detail in 3.4.2.

We will rely on these properties to assert equality of morphisms, necessary for later coherence conditions.

3.2 Category of Formulas and Proofs

We are now ready to define the Category of Formulas and Proofs, which interprets ILL with finite products with the exception of exponential modality.

Definition 1: Category of Formulas and Proofs

The *Category of Formulas and Proofs* $\mathcal{L}$ is defined as follows:

- **Objects** are denotations of ILL formulas. That is, for every formula A, there is an object $[\![A]\!]$.
- **Morphisms** are denotations of formal proofs. That is, for every proof π (a proof of $A \vdash B$), there is a morphism $[\![\pi]\!] : [\![A]\!] \to [\![B]\!]$. Notice that, by definition, every morphism in this category is invariant under cut-elimination.
- **Identity** morphism $\mathrm{id}_A : [\![A]\!] \to [\![A]\!]$ for each object A is given by the axiom rule:

$$\text{Axiom}\ \frac{}{A \vdash A}$$

- **Composition** $g \circ f : [\![A]\!] \to [\![C]\!]$ of morphisms $f : [\![A]\!] \to [\![B]\!]$ (a proof of $A \vdash B$) and $g : [\![B]\!] \to [\![C]\!]$ (a proof of $B \vdash C$) is given by the cut rule [a]:

$$\text{Cut}\ \frac{A \vdash B \qquad B \vdash C}{A \vdash C}$$

[a] Notice that composition exhibits the modularity property of proof denotations under cut-elimination.

Confirming that associativity of composition and composition of identity holds is left as an exercise.

Axiom rules are foundational to logic, so it is no wonder that this rule found its way to the base definition of the category. It's interpretation as an identity morphism means that we can always use the Axiom rule as a leaf in a proof tree.

More interestingly, the Cut rule that is only admissible but not necessary corresponds to composition. This is exactly what we would expect. By the Cut-Elimination theorem, there must exist a cut-free version of any proof, which therefore would be a non-composite morphism. Since any composite morphism $g \circ f : [\![A]\!] \to [\![C]\!]$ is understood to be a valid morphism by itself, this is in line with our expectations.

3.3 Categorifying "With" and Its Unit

Let us begin with a rather straightforward additive connective "with" (written &) and it's unit $\top$. All relevant rules are given below:

$$\text{True}\ \frac{}{\Gamma \vdash \top} \qquad \text{Right}\ \& \ \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \& B}$$

$$\text{Left}\ \&_1\ \frac{\Gamma, A \vdash C}{\Gamma, A \& B \vdash C} \qquad \text{Left}\ \&_2\ \frac{\Gamma, B \vdash C}{\Gamma, A \& B \vdash C}$$

Notice that there is no left rule for $\top$. The right rule states that we can always produce $\top$ from any context, essentially discarding all resources in Γ. $\top$ is the trivial, always-available resource that

carries no information, so it will never be "chosen" by &. It is easy to show that $A \vdash A\&\top$ and $A\&\top \vdash A$.

When interpreted as a morphism, the True rule says that for any object Γ there (axiomatically!) exists a morphism into this object $\top$. All morphisms are invariant under cut-elimination, so it must also be unique. And this is precisely the definition of a terminal object!

Now, onto the & rules. The right rule states that to produce $A\&B$, we must be able to produce both A and B from the *same* context Γ. The left rules state that when we have $A\&B$ as an assumption, we can choose to use it as either A or B, but once we make the choice, the other option is lost.

This looks a lot like the good-old categorical product. We can easily construct a product triple (P, π_1, π_2) as:

- The product P is simply the formula $A\&B$
- The first projection π_1 is the proof of $A\&B \vdash A$, which follows immediately from Left $\&_1$ rule
- The second projection π_2 is the proof of $A\&B \vdash B$, which follows immediately from Left $\&_2$ rule

With this set up, the Right & rule corresponds to the unique morphism $\langle f, g\rangle : [\![\Gamma]\!] \to [\![A\&B]\!]$ given that $f : [\![\Gamma]\!] \to [\![A]\!]$ and $g : [\![\Gamma]\!] \to [\![B]\!]$ exist. We leave confirming that the triple exhibits the universal property of products as an exercise.

With this, we learned something new about our category $\mathcal{L}$ - it has finite products.

3.4 Categorifying Tensor Product

Now for a much more fundamental connective: the multiplicative tensor product (written $\otimes$) and its unit 1. All relevant rules are given below:

$$\text{Left } 1\ \frac{\Gamma \vdash A}{\Gamma, 1 \vdash A} \qquad \text{Right } 1\ \frac{}{\vdash 1}$$

$$\text{Left } \otimes\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \qquad \text{Right } \otimes\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}$$

The Right 1 rule states that we can produce the unit from nothing (the empty context). The Left 1 rule states that having a unit resource doesn't prevent us from deriving anything else. Since 1 is a unit of a tensor product, we can show that $A \vdash A \otimes 1$ and $A \otimes 1 \vdash A$.

The Right $\otimes$ rule says that to produce $A \otimes B$, we must independently produce A from context Γ_1 and B from context Γ_2, and all resources are consumed. The Left $\otimes$ rule says that if we have $A \otimes B$ as a resource, we can unpack it (in the sense of unpacking present in sequent calculus) into separate resources A and B.

The fact that we can think of tensors on the left hand-side of the sequent simply as commas (due to Left $\otimes$) is illuminating. It suggests, that tensor product is not some composite resource like $A\&B$ was, but rather it is some structure extending the context with both resources at once.

Turns out, this is because $\otimes$ and its unit 1 make this category monoidal! The rest of this subsection will be spent demonstrating this fact.

3.4.1 Monoidal Category refresher

Recall the definition of a monoid from Chapter 11. Note that the symbol $\otimes$ is customarily used both for the linear logic connective and the bifunctor in the monoidal category. We will clarify which one we are using if necessary, but it will generally be obvious from context.

Definition 2: Monoidal Category

A category $\mathcal{C}$ is *monoidal* if:

1. there exists a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
2. there exists a distinguished object I
3. there exist three natural isomorphisms:
   - $\alpha_{A,B,C} : A \otimes (B \otimes C) \Rightarrow (A \otimes B) \otimes C$
   - $\lambda_A : I \otimes A \Rightarrow A$
   - $\rho_A : A \Rightarrow A \otimes I$
4. such that the following diagrams commute (which we'll call the "pentagonal" and "triangle" identity diagrams respectively):

$$\begin{array}{ccccc}
((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A,B,C} \otimes \mathrm{id}_D} & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A,B \otimes C,D}} & A \otimes ((B \otimes C) \otimes D) \\
\downarrow{\scriptstyle \alpha_{A \otimes B,C,D}} & & & & \downarrow{\scriptstyle \mathrm{id}_A \otimes \alpha_{B,C,D}} \\
(A \otimes B) \otimes (C \otimes D) & & \xrightarrow[\alpha_{A,B,C \otimes D}]{} & & A \otimes (B \otimes (C \otimes D))
\end{array}$$

$$\begin{array}{ccc}
(A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B) \\
& {\scriptstyle \rho_A \otimes \mathrm{id}_B} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_A \otimes \lambda_B} & \\
& A \otimes B &
\end{array}$$

Our target is to show that the tensor product can be interpreted as a bifunctor and then confirm that the rest of the conditions are consistent with ILL.

3.4.2 Tensor Product as Monoidal Bifunctor

A bifunctor is simply a functor that takes two arguments, so, just as with any other functor, we begin with defining its actions [1]:

- **Object action.** Given two objects $[\![A]\!]$ and $[\![B]\!]$, we define it as $[\![A]\!] \otimes [\![B]\!] \mapsto [\![A \otimes B]\!]$. In other words, we interpret object action on two objects $[\![A]\!]$ and $[\![B]\!]$ the bifunctor as a tensor product of formulas A and B. Such an object trivially exists from our ILL grammar.

[1] Notice that the bifunctor exhibits the tensoriality property of proof denotations under cut-elimination.

- **Morphism action.** Given two morphisms $p_1 : [\![A]\!] \to [\![B]\!]$ (proof of $A \vdash B$) and $p_2 : [\![C]\!] \to [\![D]\!]$ (proof of $C \vdash D$), we define it as $p_1 \otimes p_2 : [\![A \otimes C]\!] \to [\![B \otimes D]\!]$. This is called *parallel composition*.

  We can show that such a morphism exists and is a composite of two subproofs p_1, p_2:

$$\dfrac{\dfrac{\overset{p_1}{\vdots}}{A \vdash B} \quad \dfrac{\overset{p_2}{\vdots}}{C \vdash D}}{\dfrac{A, B \vdash B \otimes D}{A \otimes C \vdash B \otimes D}\,\text{Left} \otimes}\,\text{Right} \otimes$$

Additionally, a bifunctor must also satisfy the functoriality properties. We will first show that it respects composition. Consider four proofs:

$$p_1 : [\![A_1]\!] \to [\![A_2]\!] \qquad p_2 : [\![A_2]\!] \to [\![A_3]\!] \qquad p_3 : [\![B_1]\!] \to [\![B_2]\!] \qquad p_4 : [\![B_2]\!] \to [\![B_3]\!]$$

Then, we want to show that $(p_2 \otimes p_4) \circ (p_1 \otimes p_3)$ is equivalent to $(p_2 \circ p_1) \otimes (p_4 \circ p_3)$. Both must be valid proofs of $A_1 \otimes B_1 \vdash A_3 \otimes B_3$.

Let us derive these two proofs, knowing that Cut rule is our composition, and that the Left $\otimes$ and Right $\otimes$ rules apply the tensor product. Therefore, for $(p_2 \otimes p_4) \circ (p_1 \otimes p_3)$ we get:

$$\dfrac{\text{Right} \otimes \dfrac{\dfrac{\overset{p_1}{\vdots}}{A_1 \vdash A_2} \quad \dfrac{\overset{p_3}{\vdots}}{B_1 \vdash B_2}}{A_1, B_1 \vdash A_2 \otimes B_2}\,\text{Left} \otimes \quad \dfrac{\dfrac{\overset{p_2}{\vdots}}{A_2 \vdash A_3} \quad \dfrac{\overset{p_4}{\vdots}}{B_2 \vdash B_3}}{A_2, B_2 \vdash A_3 \otimes B_3}\,\text{Right} \otimes}{A_1 \otimes B_1 \vdash A_2 \otimes B_2 \qquad A_2 \otimes B_2 \vdash A_3 \otimes B_3}\,\text{Left} \otimes}{A_1 \otimes B_1 \vdash A_3 \otimes B_3}\,\text{Cut}$$

and for $(p_2 \circ p_1) \otimes (p_4 \circ p_3)$ we get:

$$\dfrac{\dfrac{\dfrac{\overset{p_1}{\vdots}}{A_1 \vdash A_2} \quad \dfrac{\overset{p_2}{\vdots}}{A_2 \vdash A_3}}{A_1 \vdash A_3}\,\text{Cut} \quad \dfrac{\dfrac{\overset{p_3}{\vdots}}{B_1 \vdash B_2} \quad \dfrac{\overset{p_4}{\vdots}}{B_2 \vdash B_3}}{B_1 \vdash B_3}\,\text{Cut}}{A_1, B_1 \vdash A_3 \otimes B_3}\,\text{Right} \otimes}{A_1 \otimes B_1 \vdash A_3 \otimes B_3}\,\text{Left} \otimes$$

Notice how the order in which we apply the Cut and $\otimes$ rules matches the order in which composition and tensor product are applied in the equational version. By invariance under cut-elimination, we know that the denotations of these two proofs are equal.

Moving on to the identity functoriality property. We want to prove that $id_{[\![A]\!] \otimes [\![B]\!]} = id_{[\![A]\!]} \otimes id_{[\![B]\!]}$, both of which are proofs of $A \otimes B \vdash A \otimes B$.

Similarly, we derive a tree for $id_{[\![A]\!] \otimes [\![B]\!]}$:

$$\dfrac{}{A \otimes B \vdash A \otimes B}\,\text{Axiom}$$

and a tree for $id_{[\![A]\!]} \otimes id_{[\![B]\!]}$:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\text{Axiom} \quad \dfrac{}{B \vdash B}\text{Axiom}}{A, B \vdash A \otimes B}\text{Right } \otimes}{A \otimes B \vdash A \otimes B}\text{Left } \otimes$$

By invariance under cut-elimination, we know that the denotations of these two proofs are equal. Therefore, the tensor product can indeed be interpreted as a bifunctor.

3.4.3 The 3 Natural Isomorphisms

The next step in demonstrating that tensor products make our category $\mathcal{L}$ a monoidal category is showing that the following natural isomorphisms exist:

- $\alpha_{A,B,C} : A \otimes (B \otimes C) \Rightarrow (A \otimes B) \otimes C$
- $\lambda_A : I \otimes A \Rightarrow A$
- $\rho_A : A \Rightarrow A \otimes I$

Notice that λ and ρ are defined in terms of some distinguished object I for our category, otherwise known as its *unit*. We can trivially define $I = [\![1]\!]$, the object corresponding to the tensor product's unit in the ILL rules.

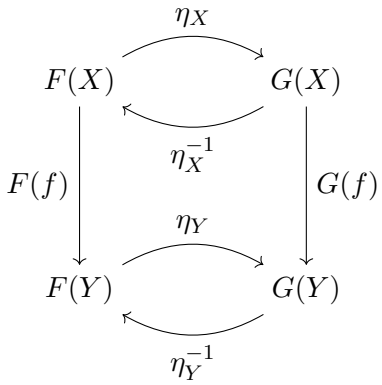
We will demonstrate how prove the existence $\alpha_{A,B,C}$ natural isomorphism indexed by objects in our category,

Now, recall the definition of a natural isomorphism.

Definition 3: Natural Isomorphism

Suppose functors $F, G : \mathcal{C} \to \mathcal{D}$ and some natural transformation $\eta : F \Rightarrow G$, η is a natural isomorphism if for every object $X \in \mathcal{C}$, the component η_X is an isomorphism.

In other words, for every $f : X \to Y \in \mathcal{C}$ the following diagram must hold:



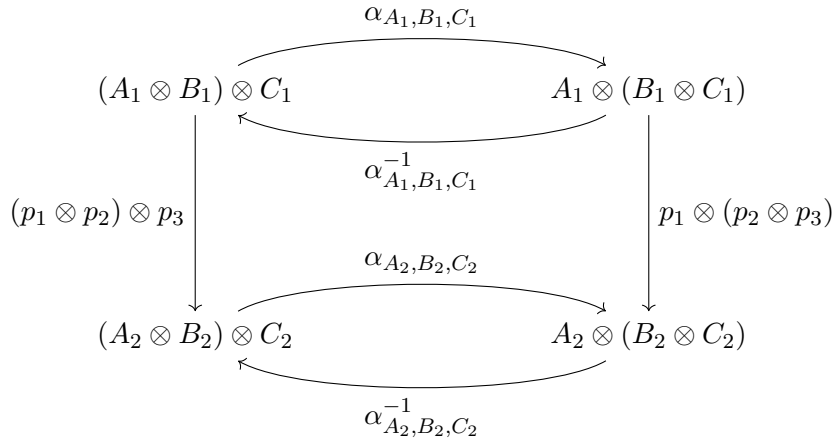
Let us consider $\alpha_{A,B,C}$ indexed by $A, B, C \in \mathcal{L}$ in detail. Instantiating the definition, η is α, F is $(\bullet_1 \otimes \bullet_2) \otimes \bullet_3 : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ and G is $\bullet_1 \otimes (\bullet_2 \otimes \bullet_3) : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$.

We also need three morphisms on six objects in $\mathcal{L}$:

$$p_1 : [\![A_1]\!] \to [\![A_2]\!] \qquad p_2 : [\![B_1]\!] \to [\![B_2]\!] \qquad p_3 : [\![C_1]\!] \to [\![C_2]\!]$$

Then, $\alpha_{A,B,C}$ is a natural iso if the following diagram commutes [2]:

$$\begin{tikzcd}
(A_1 \otimes B_1) \otimes C_1 \arrow[r, bend left, "\alpha_{A_1,B_1,C_1}"] \arrow[d, "(p_1 \otimes p_2) \otimes p_3"'] & A_1 \otimes (B_1 \otimes C_1) \arrow[l, bend left, "\alpha^{-1}_{A_1,B_1,C_1}"] \arrow[d, "p_1 \otimes (p_2 \otimes p_3)"] \\
(A_2 \otimes B_2) \otimes C_2 \arrow[r, bend left, "\alpha_{A_2,B_2,C_2}"] & A_2 \otimes (B_2 \otimes C_2) \arrow[l, bend left, "\alpha^{-1}_{A_2,B_2,C_2}"]
\end{tikzcd}$$



Now, we can demonstrate that $\alpha_{A,B,C}$ and $\alpha^{-1}_{A,B,C}$ exist and are valid. Starting with $\alpha_{A,B,C}$, we have:

$$\dfrac{\dfrac{\dfrac{\text{Axiom}\ \dfrac{}{A \vdash A} \qquad \dfrac{\text{Axiom}\ \dfrac{}{B \vdash B} \quad \dfrac{}{C \vdash C}\ \text{Axiom}}{B, C \vdash B \otimes C}\ \text{Right} \otimes}{A, B, C \vdash A \otimes (B \otimes C)}\ \text{Right} \otimes}{A \otimes B, C \vdash A \otimes (B \otimes C)}\ \text{Left} \otimes}{(A \otimes B) \otimes C \vdash A \otimes (B \otimes C)}\ \text{Left} \otimes$$

And, for $\alpha^{-1}_{A,B,C}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom}\ \dfrac{}{A \vdash A} \quad \dfrac{}{B \vdash B}\ \text{Axiom}}{A, B \vdash A \otimes B}\ \text{Right} \otimes \qquad \dfrac{}{C \vdash C}\ \text{Axiom}}{A, B, C \vdash (A \otimes B) \otimes C}\ \text{Right} \otimes}{A, B \otimes C \vdash (A \otimes B) \otimes C}\ \text{Left} \otimes}{A \otimes (B \otimes C) \vdash (A \otimes B) \otimes C}\ \text{Left} \otimes$$

Now we must show naturality of α by showing that $\alpha \circ ((p_1 \otimes p_2) \otimes p_3) = (p_1 \otimes (p_2 \otimes p_3)) \circ \alpha$. Starting off with $\alpha \circ ((p_1 \otimes p_2) \otimes p_3)$:

$$\dfrac{\text{Left} \otimes \dfrac{\text{Right} \otimes \dfrac{\dfrac{\overset{p_1}{\vdots}}{A_1 \vdash A_2} \qquad \dfrac{\dfrac{\dfrac{\overset{p_2}{\vdots}}{B_1 \vdash B_2} \quad \dfrac{\overset{p_3}{\vdots}}{C_1 \vdash C_2}}{B_1, C_1 \vdash B_2 \otimes C_2}\ \text{Right} \otimes}{B_1 \otimes C_1 \vdash B_2 \otimes C_2}\ \text{Left} \otimes}{A_1, B_1 \otimes C_1 \vdash A_2 \otimes (B_2 \otimes C_2)}}{A_1 \otimes (B_1 \otimes C_1) \vdash A_2 \otimes (B_2 \otimes C_2)} \qquad \dfrac{\overset{\alpha}{\vdots}}{A_2 \otimes (B_2 \otimes C_2) \vdash (A_2 \otimes B_2) \otimes C_2}}{A_1 \otimes (B_1 \otimes C_1) \vdash (A_2 \otimes B_2) \otimes C_2}\ \text{Cut}$$

and for $(p_1 \otimes (p_2 \otimes p_3)) \circ \alpha$:

[2] Note: At this point we will stop distinguishing between formula A and its denotation $[\![A]\!]$ in commutative diagrams

$$\dfrac{\dfrac{\overset{\alpha}{\vdots}}{(A_1 \otimes B_1) \otimes C_1 \vdash A_1 \otimes (B_1 \otimes C_1)} \quad \dfrac{\dfrac{\overset{p_1}{\vdots}}{A_1 \vdash A_2} \quad \dfrac{\dfrac{\overset{p_2}{\vdots}}{B_1 \vdash B_2} \quad \dfrac{\overset{p_3}{\vdots}}{C_1 \vdash C_2}}{B_1, C_1 \vdash B_2 \otimes C_2}\,\text{Right}\otimes}{B_1 \otimes C_1 \vdash B_2 \otimes C_2}\,\text{Left}\otimes}{A_1, B_1 \otimes C_1 \vdash A_2 \otimes (B_2 \otimes C_2)}\,\text{Right}\otimes}{A_1 \otimes (B_1 \otimes C_1) \vdash A_2 \otimes (B_2 \otimes C_2)}\,\text{Left}\otimes$$
$$(A_1 \otimes B_1) \otimes C_1 \vdash A_2 \otimes (B_2 \otimes C_2)\,\text{Cut}$$

By invariance and tensoriality under cut-elimination these two trees are equivalent, so we have shown that $\alpha \circ ((p_1 \otimes p_2) \otimes p_3) = (p_1 \otimes (p_2 \otimes p_3)) \circ \alpha$. Now to finish showing that α is a natural isomorphism we must show that α and its inverse round trip to identity. This is a simple application of the cut rule. For $\alpha \circ \alpha^{-1} = id_{A \otimes (B \otimes C)}$ and $\alpha^{-1} \circ \alpha = id_{(A \otimes B) \otimes C}$ we have:

$$\dfrac{\dfrac{\overset{\alpha}{\vdots}}{A \otimes (B \otimes C) \vdash (A \otimes B) \otimes C} \quad \dfrac{\overset{\alpha^{-1}}{\vdots}}{(A \otimes B) \otimes C \vdash A \otimes (B \otimes C)}}{A \otimes (B \otimes C) \vdash A \otimes (B \otimes C)}\,\text{Cut}$$

$$\dfrac{\dfrac{\overset{\alpha^{-1}}{\vdots}}{(A \otimes B) \otimes C \vdash A \otimes (B \otimes C)} \quad \dfrac{\overset{\alpha}{\vdots}}{A \otimes (B \otimes C) \vdash (A \otimes B) \otimes C}}{(A \otimes B) \otimes C \vdash (A \otimes B) \otimes C}\,\text{Cut}$$

Both these equations have an immediate axiomatic derivation for the identity natural transformations on the right hand-side. For both of them, by tensoriality and invariance under cut-elimination, these equalities hold. With this we have shown that α is a natural isomorphism.

Proofs for λ and ρ go similarly. These are derivations for existence of λ_A and λ_A^{-1} respectively:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{Axiom}}{1, A \vdash A}\,\text{Left 1}}{1 \otimes A \vdash A}\,\text{Left}\otimes \qquad \text{Right 1}\,\dfrac{\dfrac{}{\vdash 1} \quad \dfrac{\dfrac{}{A \vdash A}\,\text{Axiom}}{A \vdash 1 \otimes A}\,\text{Right}\otimes}{}$$

And these are derivations for existence of ρ_A and ρ_A^{-1} respectively:

$$\text{Axiom}\,\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{\vdash 1}\,\text{Right 1}}{A \vdash A \otimes 1}\,\text{Right}\otimes \qquad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{Axiom}}{A, 1 \vdash A}\,\text{Left 1}}{A \otimes 1 \vdash A}\,\text{Left}\otimes$$

It is straightforward to demonstrate naturality and isomorphism from here.[3] One interesting observation is that derivations for λ and ρ are mirror images of each other, so we can define one in terms of another relying on the Exchange rule.

With this, we have demonstrated that $\mathcal{L}$ has the three natural isomorphisms α, λ and ρ.

[3] Mellies remarks that a conventional cut-elimination procedure would not enforce equality of denotations in some edge cases (η-expansion of the identity), but the procedure he defines in Chapter 3 addresses this issue.[9]

3.4.4 The Pentagonal and Triangle Identities

The last thing we need is to demonstrate that the pentagonal and triangle identities hold. Let's start with the pentagonal identity.

$$\begin{array}{ccccc}
((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A,B,C} \otimes \mathrm{id}_D} & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A,B \otimes C,D}} & A \otimes ((B \otimes C) \otimes D) \\
\Big\downarrow{\scriptstyle \alpha_{A \otimes B,C,D}} & & & & \Big\downarrow{\scriptstyle \mathrm{id}_A \otimes \alpha_{B,C,D}} \\
(A \otimes B) \otimes (C \otimes D) & & \xrightarrow[\alpha_{A,B,C \otimes D}]{} & & A \otimes (B \otimes (C \otimes D))
\end{array}$$

In these derivation trees we will rely on derivations we did earlier for α, λ and ρ as subproofs. First, we show $\alpha_{A,B,C} \otimes id_D : ((A \otimes B) \otimes C) \otimes D \to (A \otimes (B \otimes C)) \otimes D$:

$$\dfrac{\dfrac{\dfrac{\overset{\alpha}{\vdots}}{(A \otimes B) \otimes C \vdash A \otimes (B \otimes C)} \quad \dfrac{}{D \vdash D}\,\text{Axiom}}{((A \otimes B) \otimes C), D \vdash (A \otimes (B \otimes C)) \otimes D}\,\text{Right} \otimes}{((A \otimes B) \otimes C) \otimes D \vdash (A \otimes (B \otimes C)) \otimes D}\,\text{Left} \otimes$$

Next, we show $\alpha_{A,B \otimes C,D} : (A \otimes (B \otimes C)) \otimes D \to A \otimes ((B \otimes C) \otimes D)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{Axiom} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,\text{Axiom} \quad \dfrac{}{C \vdash C}\,\text{Axiom}}{B, C \vdash B \otimes C}\,\text{Right} \otimes}{B \otimes C \vdash B \otimes C}\,\text{Left} \otimes \quad \dfrac{}{D \vdash D}\,\text{Axiom}}{B \otimes C, D \vdash (B \otimes C) \otimes D}\,\text{Right} \otimes}{A, B \otimes C, D \vdash A \otimes ((B \otimes C) \otimes D)}\,\text{Right} \otimes}{A \otimes (B \otimes C), D \vdash A \otimes ((B \otimes C) \otimes D)}\,\text{Left} \otimes}{(A \otimes (B \otimes C)) \otimes D \vdash A \otimes ((B \otimes C) \otimes D)}\,\text{Left} \otimes$$

Next, we show $id_A \otimes \alpha_{B,C,D} : A \otimes ((B \otimes C) \otimes D) \to A \otimes (B \otimes (C \otimes D))$:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{Axiom} \quad \dfrac{\overset{\alpha}{\vdots}}{(B \otimes C) \otimes D \vdash B \otimes (C \otimes D)}}{A, (B \otimes C) \otimes D \vdash A \otimes (B \otimes (C \otimes D))}\,\text{Right} \otimes}{A \otimes ((B \otimes C) \otimes D) \vdash A \otimes (B \otimes (C \otimes D))}\,\text{Left} \otimes$$

With this we've completed one path through the pentagonal identity $p_1 = (id_A \otimes \alpha_{B,C,D}) \circ (\alpha_{A,B \otimes C,D}) \circ (\alpha_{A,B,C} \otimes id_D)$. Now, let's complete the second path starting with $\alpha_{A \otimes B,C,D} : ((A \otimes B) \otimes C) \otimes D \to (A \otimes B) \otimes (C \otimes D)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{Axiom} \quad \dfrac{}{B \vdash B}\,\text{Axiom}}{A, B \vdash A \otimes B}\,\text{Right} \otimes}{A \otimes B \vdash A \otimes B}\,\text{Left} \otimes \quad \dfrac{\dfrac{}{C \vdash C}\,\text{Axiom} \quad \dfrac{}{D \vdash D}\,\text{Axiom}}{C, D \vdash C \otimes D}\,\text{Right} \otimes}{A \otimes B, C, D \vdash (A \otimes B) \otimes (C \otimes D)}\,\text{Right} \otimes}{(A \otimes B) \otimes C, D \vdash (A \otimes B) \otimes (C \otimes D)}\,\text{Left} \otimes}{((A \otimes B) \otimes C) \otimes D \vdash (A \otimes B) \otimes (C \otimes D)}\,\text{Left} \otimes$$

And this is the last morphism $\alpha_{A,B,C\otimes D} : (A \otimes B) \otimes (C \otimes D) \to A \otimes (B \otimes (C \otimes D))$:

$$\dfrac{\dfrac{\text{Axiom}\ \dfrac{}{A \vdash A} \qquad \dfrac{\dfrac{\text{Axiom}\ \dfrac{}{B \vdash B} \qquad \dfrac{\dfrac{\text{Axiom}\ \dfrac{}{C \vdash C} \qquad \text{Axiom}\ \dfrac{}{D \vdash D}}{C, D \vdash C \otimes D}\ \text{Right}\otimes}{C \otimes D \vdash C \otimes D}\ \text{Left}\otimes}{B, C \otimes D \vdash B \otimes (C \otimes D)}\ \text{Right}\otimes}{A, B, C \otimes D \vdash A \otimes (B \otimes (C \otimes D))}\ \text{Right}\otimes}{A \otimes B, C \otimes D \vdash A \otimes (B \otimes (C \otimes D))}\ \text{Left}\otimes}{(A \otimes B) \otimes (C \otimes D) \vdash A \otimes (B \otimes (C \otimes D))}\ \text{Left}\otimes$$

This completes the second path pentagonal identity $p_2 = (\alpha_{A,B,C\otimes D}) \circ (\alpha_{A\otimes B,C,D})$. Now we must show that $p_1 = p_2$, starting with p_1 we have:

$$\dfrac{\dfrac{\dfrac{\overset{\alpha_{A,B,C}\otimes id_D}{\vdots}}{((A \otimes B) \otimes C) \otimes D \vdash (A \otimes (B \otimes C)) \otimes D} \qquad \dfrac{\dfrac{\overset{\alpha_{A,B\otimes C,D}}{\vdots}}{(A \otimes (B \otimes C)) \otimes D \vdash A \otimes ((B \otimes C) \otimes D)} \qquad \dfrac{\overset{id_A\otimes\alpha_{B,C,D}}{\vdots}}{A \otimes ((B \otimes C) \otimes D) \vdash A \otimes (B \otimes (C \otimes D))}\ \text{Cut}}{(A \otimes (B \otimes C)) \otimes D \vdash A \otimes (B \otimes (C \otimes D))}\ \text{Cut}}{((A \otimes B) \otimes C) \otimes D \vdash A \otimes (B \otimes (C \otimes D))}$$

and now for p_2:

$$\dfrac{\dfrac{\overset{\alpha_{A\otimes B,C,D}}{\vdots}}{((A \otimes B) \otimes C) \otimes D \vdash (A \otimes B) \otimes (C \otimes D)} \qquad \dfrac{\overset{\alpha_{A,B,C\otimes D}}{\vdots}}{(A \otimes B) \otimes (C \otimes D) \vdash A \otimes (B \otimes (C \otimes D))}}{((A \otimes B) \otimes C) \otimes D \vdash A \otimes (B \otimes (C \otimes D))}\ \text{Cut}$$

By tesnosiality and invariance under cut-elimination these two paths are equal, so we've shown that the pentagonal identity holds. Now for the triangle identity. Note that we replaced distinguished object I with it's interpretation 1.

$$\begin{array}{ccc} (A \otimes 1) \otimes B & \xrightarrow{\qquad\qquad \alpha_{A,1,B} \qquad\qquad} & A \otimes (1 \otimes B) \\ & {\scriptstyle \rho_A \otimes \mathrm{id}_B} \searrow \qquad \swarrow {\scriptstyle \mathrm{id}_A \otimes \lambda_B} & \\ & A \otimes B & \end{array}$$

We first show $\alpha_{A,1,B} : (A \otimes I) \otimes B \to A \otimes (1 \otimes B)$:

$$\dfrac{\dfrac{\dfrac{\text{Axiom}\ \dfrac{}{A \vdash A} \qquad \dfrac{\text{Right}\ 1\ \dfrac{}{\vdash 1} \qquad \dfrac{\text{Axiom}\ \dfrac{}{B \vdash B}}{1, B \vdash B}\ \text{Left}\ 1}{1, B \vdash 1 \otimes B}\ \text{Right}\otimes}{A, 1, B \vdash A \otimes (1 \otimes B)}\ \text{Right}\otimes}{A \otimes 1, B \vdash A \otimes (1 \otimes B)}\ \text{Left}\otimes}{(A \otimes 1) \otimes B \vdash A \otimes (1 \otimes B)}\ \text{Left}\otimes$$

Next, we show $id_A \otimes \lambda_B : A \otimes (1 \otimes B) \to A \otimes B$:

$$\dfrac{\text{Axiom}\ \dfrac{}{A \vdash A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\ \text{Axiom}}{1, B \vdash B}\ \text{Left 1}}{1 \otimes B \vdash B}\ \text{Left} \otimes}{A, 1 \otimes B \vdash A \otimes B}\ \text{Right} \otimes}{A \otimes (1 \otimes B) \vdash A \otimes B}\ \text{Left} \otimes$$

And, lastly, for short path, we show $\rho_A \otimes id_B : (A \otimes 1) \otimes B \to A \otimes B$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\ \text{Axiom}}{A, 1 \vdash A}\ \text{Left 1}}{A \otimes 1 \vdash A}\ \text{Left} \otimes \qquad \dfrac{}{B \vdash B}\ \text{Axiom}}{A \otimes 1, B \vdash A \otimes B}\ \text{Right} \otimes}{(A \otimes 1) \otimes B \vdash A \otimes B}\ \text{Left} \otimes$$

To show that the diagram commutes, we need to show that $(id_A \otimes \lambda_B) \circ (\alpha_{A,1,B}) = \rho_A \otimes id_B$. Starting off with $(id_A \otimes \lambda_B) \circ (\alpha_{A,1,B})$:

$$\dfrac{\dfrac{\overset{\alpha_{A,1,B}}{\vdots}}{(A \otimes 1) \otimes B \vdash A \otimes (1 \otimes B)} \qquad \dfrac{\overset{id_A \otimes \lambda_B}{\vdots}}{A \otimes (1 \otimes B) \vdash A \otimes B}}{(A \otimes 1) \otimes B \vdash A \otimes B}\ \text{Cut}$$

We have already shown the derivation tree for $\rho_A \otimes id_B$, and by tensoriality and invariance under cut-elimination these two trees are equivalent. Therefore we have shown the triangle identity holds and our category is monoidal.

3.5 Categorifying Linear Implication

Next up is linear implication connective $\multimap$ with it's pair of rules:

$$\text{Left} \multimap \dfrac{\Gamma_1 \vdash A \qquad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \multimap B \vdash C} \qquad\qquad \text{Right} \multimap \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$$

Right $\multimap$ rule asserts that if we have a proof of $\Gamma, A \vdash B$ (corresponding categorically to a morphism $\Gamma \otimes A \to B$), then we can construct a proof of $\Gamma \vdash A \multimap B$. The Left $\multimap$ rule tells us that if we have a proof of $\Gamma_1 \vdash A$ and $\Gamma, B \vdash C$, we are allowed to introduce a new assumption of $A \multimap B$.

Intuitively, an implication $A \multimap B$ consumes a formula A and turns it into a formula B. This feels really similar to the sequent $A \vdash B$, but there is an important difference: the proof $A \vdash B$ corresponds to a *morphism* in our category, while $A \multimap B$ is an *object*. How can we represent this "object of morphisms" in our category?

The answer is, of course, the *exponential object* B^A, which represents "all functions from A to B" in categories with finite products. However, in the setting of linear logic, we want a similar construction that works with the tensor product $\otimes$ rather than the cartesian product $\times$. Such a construction is called an *internal hom* or *closed structure*, and a monoidal category equipped with this structure is called *monoidal closed*. [4]

[4] In fact, the exponential object in a cartesian closed category is is simply a special case of the internal hom in a monoidal category where the monoidal product is the cartesian product $\times$ and the unit is the terminal object.

3.5.1 Monoidal Closed Category

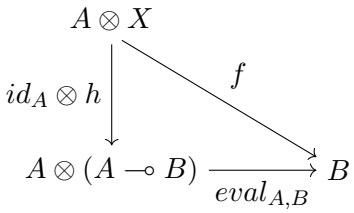
Let us be more precise with the definition.

Definition 4: Monoidal Closed Category

A monoidal category $(L, \otimes, 1)$ is *monoidal closed* if for any two objects A and B, there exist a *left closed structure*:

- an object $A \multimap B$ (the internal hom)
- a morphism $\text{eval}_{A,B} : A \otimes (A \multimap B) \to B$ (the left evaluation morphism)

such that for every morphism $f : A \otimes X \to B$, there exists a unique morphism $h : X \to A \multimap B$ such that the following diagram commutes:



Interestingly, this universal property establishes a natural bijection

$$\text{Hom}(A \otimes X, B) \cong \text{Hom}(X, A \multimap B)$$

which is precisely the statement that $\otimes$ and $\multimap$ form an adjunction (with $\otimes$ left adjoint to $\multimap$). Showing that this bijection indeed asserts that linear implication is an internal hom of our category is left as an exercise.

The morphism h is often called the *transpose* or *currying* of f.

3.5.2 Linear Implication as Internal Hom

Given our logic rules, we can construct a left closed structure in category $\mathcal{L}$. Let linear implication $A \multimap B$ be the internal hom. Then $\text{eval}_{A,B}$ exists by construction:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{Axiom} \quad \dfrac{}{B \vdash B}\,\text{Axiom}}{A, A \multimap B \vdash B}\,\text{Left} \multimap}{A \otimes (A \multimap B) \vdash B}\,\text{Left} \otimes$$

This derivation shows us that the Left $\multimap$ rule corresponds to the $eval_{A,B}$ morphism for any two objects A and B. This can also be seen for shape of the sequent.

Next, we demonstrate that the universal property holds. Suppose we have some morphism f given proof of $A, X \vdash B$ (equivalently, $A \otimes X \vdash B$ after applying Left $\otimes$ rule). Then we should be able to construct some proof of $X \vdash A \multimap B$, we will call this morphism h.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}f\\ \vdots\\ A, X \vdash B\end{array}}{X, A \vdash B}\,\text{Exchange}}{X \vdash A \multimap B}\,\text{Right } \multimap}$$

By invariance and tensorialty under cut-elimination, we know that this is a unique morphism. Notice how the Right $\multimap$ is precisely the currying operation guaranteed by the universal property of the internal hom. Now that we know this morphism exists, we need to show that the diagram commutes.

$$\dfrac{\dfrac{\dfrac{\text{Axiom}\,\dfrac{}{A \vdash A} \qquad \dfrac{\dfrac{\begin{array}{c}f\\ \vdots\\ A, X \vdash B\end{array}}{X, A \vdash B}\,\text{Exchange}}{X \vdash A \multimap B}\,\text{Right } \multimap}{A, X \vdash A \otimes (A \multimap B)}\,\text{Right } \otimes}{A \otimes X \vdash A \otimes (A \multimap B)}\,\text{Left } \otimes$$

With both the transpose and evaluation morphisms established, and assuming $f : A \otimes X \to B$ exists, the commuting diagram follows from the composition of these derivations via the cut rule. By the cut-elimination theorem, this composition reduces to the original proof f, confirming that the universal property is satisfied.

We have shown that category $\mathcal{L}$ has a left closed structure in, and, therefore, category $\mathcal{L}$ is a *closed monoidal category*.

3.5.3 Symmetry in Monoidal Categories

The last structural rule of sequent calculus that have not yet examined in detail is *exchange rule*, and it turns our relevant for demonstrating one more property of the category $\mathcal{L}$. The exchange rule is stated as follows:

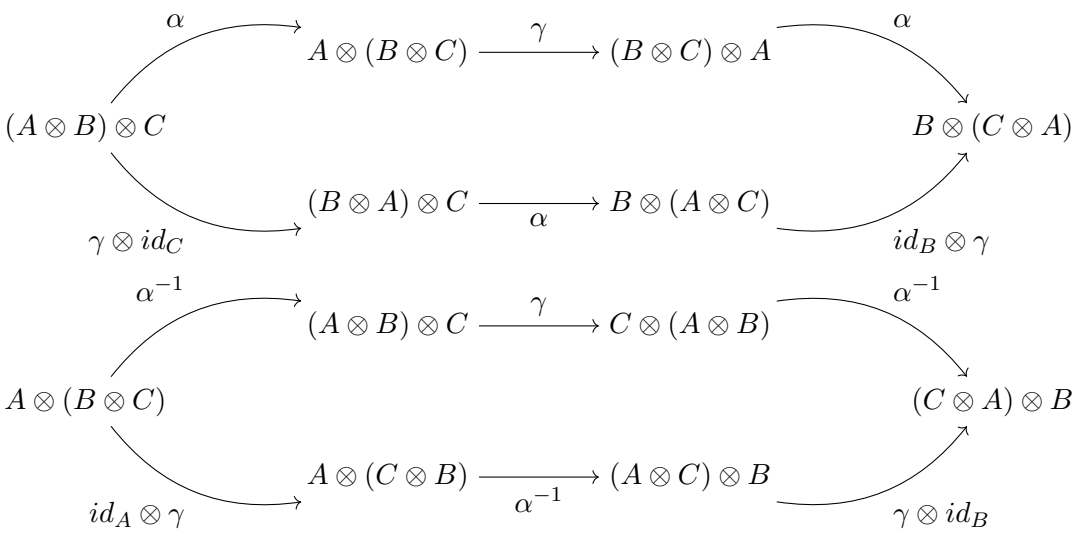
$$\text{Exchange}\,\frac{\Gamma, A, B \vdash C}{\Gamma, B, A \vdash C}$$

Setting aside the unchanged context Γ, this rule tells us that $A \otimes B$ and $B \otimes A$ are interchangeable. In other words, the exchange rule codifies the commutativity of the tensor product.

Formally, this commutativity is captured by a natural isomorphism

$$\gamma_{A,B} : A \otimes B \Rightarrow B \otimes A$$

called a *braiding*. A valid braiding will make the following two diagrams commute for all objects A, B, C:



Here, α and α^{-1} denote the associativity isomorphisms that are part of the monoidal structure. We leave the verification that the exchange rule in sequent calculus ensures these diagrams commute as an exercise.

A braided monoidal category is called *symmetric* if the braiding satisfies an additional condition: for all objects A and B,

$$\gamma_{B,A} = \gamma_{A,B}^{-1}$$

In other words, the natural isomorphism $A \otimes B \Rightarrow B \otimes A$ and its inverse $B \otimes A \Rightarrow A \otimes B$ are mutual inverses. The Exchange rule straightforwardly gives us precisely this property, making our tensor product is fully commutative. This allows us to treat the context Γ on the left-hand side of a sequent as an unordered collection of resources.

As a result, our category $\mathcal{L}$ is symmetric.

3.6 Properties of Category of Formulas and Proofs

Having interpreted everything except the exponential modality in ILL with finite products, we can now see how $\mathcal{L}$ has enough structure to support pretty much all resource-sensitive structure.

Before we look into how to escape linearity, let us summarize what we know about our category thus far:

- $\mathcal{L}$ has finite products (see 3.3)
- $\mathcal{L}$ is a monoidal category (see 3.4)
- $\mathcal{L}$ is a symmetric monoidal closed category (SMCC) (see 3.5.2 and 3.5.3)

Now, for the fun part.

4 Categorifying Exponential Modality

4.1 Exponential Modality as Comonad (Almost)

The final set of sequent calculus rules we must examine are those governing the exponential modality !, which provides controlled versions of the structural rules that were removed from classical logic to obtain linear logic. These rules are:

$$\text{Promotion } \frac{!\Gamma \vdash A}{!\Gamma \vdash !A} \qquad \text{Dereliction } \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}$$

$$\text{Weakening } \frac{\Gamma \vdash B}{\Gamma, !A \vdash B} \qquad \text{Contraction } \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$$

These four rules demonstrate that the exponential modality ! allows us to escape the linear constraints in a controlled manner, enabling us to duplicate and discard resources marked with !.

This pattern of wrapping certain values in a special context that grants them additional capabilities is very reminiscent of monads. Just as a monad allows us to work with values in a computational context (handling effects like state, exceptions), the exponential modality ! provides a principled way to work with resources that behave non-linearly.

However, there's a difference: while monads are about "wrapping" outputs, exponential modality rules like Dereliction hint that we are actually interested in "unwrapping". It turns out that we can define ! as a dual of a monad, something called a *comonad*.

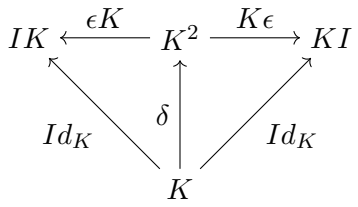
Definition 5: Comonad

A *comonad* on a category $\mathcal{L}$ is defined as a triple (K, δ, ϵ) consisting of:

- An endofunctor $K : \mathcal{L} \to \mathcal{L}$,
- A natural transformation $\delta : K \Rightarrow K \circ K$ (called the *comultiplication*),
- A natural transformation $\epsilon : K \Rightarrow \mathrm{Id}_{\mathcal{L}}$ (called the *counit*),

subject to the requirement that the following coherence diagrams commute for all objects in $\mathcal{L}$:

$$\begin{array}{ccc} K & \xrightarrow{\delta} & K^2 \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle K\delta} \\ K^2 & \xrightarrow[\delta K]{} & K^3 \end{array}$$



In our case, the endofunctor is the exponential modality !, yielding the comonad $(!, \delta, \epsilon)$. To verify that this constitutes a genuine comonad, we must demonstrate that δ and ϵ exist as natural transformations and that they satisfy the required coherence conditions. We begin by establishing the correspondence between the sequent calculus rules and the categorical structure.

For the comultiplication $\delta :! \Rightarrow !!$, we require a morphism $!A \rightarrow !!A$ for each object A. Similarly, for the counit $\epsilon :! \Rightarrow \mathrm{Id}$, we require a morphism $!A \rightarrow A$ for each object A. We can easily construct such diagrams with the following derivations:

$$\dfrac{\dfrac{}{!A \vdash !A}\,\text{Axiom}}{!A \vdash !!A}\,\text{Promotion} \qquad\qquad \dfrac{\dfrac{}{A \vdash A}\,\text{Axiom}}{!A \vdash A}\,\text{Dereliction}$$

We can see that for δ we had to use the Promotion rule, and for ϵ we had to use the Dereliction rule, showing us that these rules correspond to the comonad's structure.

To complete the verification that $(!, \delta, \epsilon)$ forms a comonad, we must prove that the coherence diagrams commute. We establish *coassociativity* by showing that the morphisms $\delta_{!A} \circ \delta_A :!A \rightarrow !!!A$ and $!(\delta_A) \circ \delta_A :!A \rightarrow !!!A$ are equal. The corresponding derivation trees are:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{!A \vdash !A}\,\text{Axiom}}{!A \vdash !!A}\,\text{Promotion} \qquad \dfrac{\dfrac{}{!!A \vdash !!A}\,\text{Axiom}}{!!A \vdash !!!A}\,\text{Promotion}}{!A \vdash !!!A}\,\text{Cut}$$

$$\dfrac{\dfrac{\dfrac{}{!A \vdash !A}\,\text{Axiom}}{!A \vdash !!A}\,\text{Promotion} \qquad \dfrac{\dfrac{}{!!A \vdash !!A}\,\text{Axiom} \qquad \dfrac{\dfrac{}{!!A \vdash !!A}\,\text{Axiom}}{!!A \vdash !!!A}\,\text{Promotion}}{!!A \vdash !!!A}\,\text{Cut}}{!A \vdash !!!A}\,\text{Cut}$$

By the cut-elimination theorem, these two derivation trees reduce to the same cut-free normal form under elimination of cuts. Therefore, $\delta_{!A} \circ \delta_A = !(\delta_A) \circ \delta_A$, establishing that the coassociativity diagram commutes.

Now, for *counitality*, we must show that $\varepsilon_{!A} \circ \delta_A = \mathrm{id}_{!A}$ and $!(\varepsilon_A) \circ \delta_A = \mathrm{id}_{!A}$. The corresponding derivation trees are:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{!A \vdash !A}\,\text{Axiom}}{!A \vdash !!A}\,\text{Promotion} \qquad \dfrac{\dfrac{}{!A \vdash !A}\,\text{Axiom}}{!!A \vdash !A}\,\text{Dereliction}}{!A \vdash !A}\,\text{Cut}$$

$$\dfrac{\dfrac{\dfrac{}{!A \vdash !A}\,\text{Axiom}}{!A \vdash !!A}\,\text{Promotion} \qquad \dfrac{\dfrac{\dfrac{}{!A \vdash !A}\,\text{Axiom}}{!!A \vdash !A}\,\text{Dereliction} \qquad \dfrac{}{!A \vdash !A}\,\text{Axiom}}{!!A \vdash !A}\,\text{Cut}}{!A \vdash !A}\,\text{Cut}$$

Once again, by cut-elimination theorem, both of these derivation trees reduce to the identity proof on $!A$. Thus, $\varepsilon_{!A} \circ \delta_A = !(\varepsilon_A) \circ \delta_A = \mathrm{id}_{!A}$, confirming that the counitality diagram commutes.

We have thus established that the exponential modality !, together with the natural transformations δ and ϵ derived from the Promotion and Dereliction rules, forms a comonad on our category. However, this is not yet sufficient for the categorical semantics of ILL. As Bierman [2] observes: "...we need some more structure. If Γ is represented by the object $A_1 \otimes \cdots \otimes A_n$, then $!\Gamma$ will be represented by $!(A_1 \otimes \cdots \otimes A_n)$, but clearly by $!\Gamma$ we really mean $!A_1 \otimes \cdots \otimes !A_n$."

To bridge this gap, we require that ! be not merely a comonad, but a *symmetric monoidal comonad*. This, demands that ! is also equipped with structure maps:

- A natural transformation $m_{A,B} : !A \otimes !B \Rightarrow !(A \otimes B)$ for all objects A, B,
- A morphism $m_1 : 1 \to !1$ (where 1 is the monoidal unit).

We can construct them as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{Axiom}}{!A \vdash A}\,\text{Dereliction} \quad \dfrac{\dfrac{}{B \vdash B}\,\text{Axiom}}{!B \vdash B}\,\text{Dereliction}}{!A, !B \vdash A \otimes B}\,\text{Right } \otimes}{!A, !B \vdash !(A \otimes B)}\,\text{Promotion}}{!A \otimes !B \vdash !(A \otimes B)}\,\text{Left } \otimes \qquad \dfrac{\dfrac{\dfrac{}{\vdash 1}\,\text{Right 1}}{\vdash !1}\,\text{Promotion}}{1 \vdash !1}\,\text{Left 1}$$

Furthermore, the symmetric monoidal structure must be compatible with the monoidal structure of the category itself. Specifically, ! must respect the associativity isomorphism α, the left and right λ and ρ, and the symmetry (braiding) γ of the underlying symmetric monoidal category. This compatibility is expressed by the commutativity of the following diagrams:

$$\begin{array}{ccc} !1 \otimes !A & \xrightarrow{m_{1,A}} & !(1 \otimes A) \\ {\scriptstyle m_1 \otimes id_{!A}}\downarrow & & \downarrow{\scriptstyle !(\lambda_A)} \\ I \otimes !A & \xrightarrow[\lambda_{!A}]{} & !A \end{array}$$

$$\begin{array}{ccc} !A \otimes !1 & \xrightarrow{m_{A,1}} & !(A \otimes 1) \\ {\scriptstyle id_{!A} \otimes m_1}\downarrow & & \downarrow{\scriptstyle !(\rho_A)} \\ !A \otimes 1 & \xrightarrow[\rho_{!A}]{} & !A \end{array}$$

$$\begin{array}{ccccc} (!A \otimes !B) \otimes !C & \xrightarrow{m_{A,B} \otimes id_{!C}} & !(A \otimes B) \otimes !C & \xrightarrow{m_{A \otimes B, C}} & !((A \otimes B) \otimes C) \\ {\scriptstyle \alpha_{!A,!B,!C}}\downarrow & & & & \downarrow{\scriptstyle !(\alpha_{A,B,C})} \\ !A \otimes (!B \otimes !C) & \xrightarrow[id_{!A} \otimes m_{B,C}]{} & !A \otimes !(B \otimes C) & \xrightarrow[m_{A, B \otimes C}]{} & !A \otimes (B \otimes C) \end{array}$$

$$\begin{array}{ccc} !A \otimes !B & \xrightarrow{m_{A,B}} & !(A \otimes B) \\ {\scriptstyle \gamma_{A,B}}\downarrow & & \downarrow{\scriptstyle !(\gamma_{A,B})} \\ !B \otimes !A & \xrightarrow[m_{B,A}]{} & !(B \otimes A) \end{array}$$

At this point, providing explicit derivation trees for each of these coherence conditions would be both tedious and, frankly, uninteresting. The interested reader is encouraged to verify these diagrams by constructing the appropriate proofs in the sequent calculus and appealing to the cut-elimination theorem to establish equality.

We've shown the existence of the comonad $(!, \delta, \epsilon)$ in which Promotion and Dereliction rules correspond to the comultiplication δ and counit ϵ, respectively. However, we are yet to account for the Contraction and Weakening rules, which are, frankly, much more fundamental.

Recall that linear logic is defined by the absence of these two structural rules, and comonad semantics of exponential modality clearly do not let us use these rules. Still, we are on the right track. To see how we can extract more structure from the ! comonad, we'll first have talk about *coalgebras*.

4.2 Coalgebras Inside Our Category

We begin by defining what it means for an object to be a coalgebra with respect to our comonad !.

Definition 6: !-Coalgebra

Given δ_A and ε_A as the comultiplication and counit components of comonad $(!, \delta, \epsilon)$, a !-*coalgebra* is a pair (A, h_A) where

- A is an object in our category $\mathcal{L}$
- $h_A : A \to !A$ is a morphism (called the *structure map*) such that these Coassociativity and Counitality diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{h_A} & !A \\ {\scriptstyle h_A}\downarrow & & \downarrow{\scriptstyle !h_A} \\ !A & \xrightarrow[\delta_A]{} & !!A \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{h_A} & !A \\ & \searrow & \downarrow{\scriptstyle \epsilon_A} \\ & & A \end{array}$$

a

[a]Given a comonad $(!, \delta, \epsilon)$ on a category $\mathcal{L}$, we can form the *category of coalgebras*, denoted $\mathcal{L}^!$, whose objects are !-coalgebras and whose morphisms are coalgebra morphisms (defined below). The dual construction for monads yields what is known as an Eilenberg-Moore category.

We will also define what it means to have a coalgebra morphism.

Definition 7: !-coalgebra morphism

A !-*coalgebra morphism* from (A, h_A) to (B, h_B) is defined as a morphism $f : A \to B$ in $\mathcal{L}$ such that $!f \circ h_A = h_B \circ f$.

Diagrammatically, this means the following square commutes:

$$\begin{array}{ccc} A & \xrightarrow{h_A} & !A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle !f} \\ B & \xrightarrow[h_B]{} & !B \end{array}$$

Now that we have established what a !-coalgebra is[5], let's see some examples in our category that will be necessary for a later proof.

Proposition 1: Given a comonad $(!, \delta, \epsilon)$ on a symmetric monoidal category $\mathcal{L}$, the pair $(!A, \delta_A : !A \to !!A))$ is a !-coalgebra for any object A.

Proof: We must verify the coassociativity and counitality axioms. The required diagrams commute directly by the coherence conditions defining a comonad:

$$\begin{array}{ccc} !A & \xrightarrow{\delta_A} & !!A \\ {\scriptstyle \delta_A}\downarrow & & \downarrow{\scriptstyle !\delta_A} \\ !!A & \xrightarrow[\delta_{!A}]{} & !!!A \end{array}$$

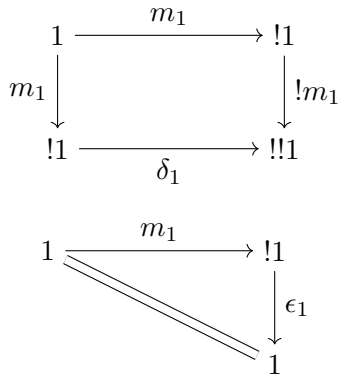
$$\begin{array}{ccc} !A & \xrightarrow{\delta_A} & !!A \\ & \searrow & \downarrow{\scriptstyle \epsilon_{!A}} \\ & & !A \end{array}$$

□

Proposition 2: Given a monoidal comonad $(!, \delta, \epsilon, m_{A,B}, m_1)$ on a symmetric monoidal category $\mathcal{L}$, the pair $(1, m_1 : 1 \to !A)$ is a !-coalgebra.

[5]An alternative perspective on these coalgebras comes from the observation that the forgetful functor $U : \mathcal{L}^! \to \mathcal{L}$, defined by $(A, h_A) \mapsto A$ on objects and $f \mapsto f$ on morphisms, admits a right adjoint (the *free functor*) $F : \mathcal{L} \to \mathcal{L}^!$ defined by $F(A) = (!A, \delta_A : !A \to !!A)$. See Lafont Catefories for more.

Proof: The coassociativity and counitality axioms follow from the fact that δ and ε are monoidal natural transformations, which ensures that they interact appropriately with the monoidal unit structure:

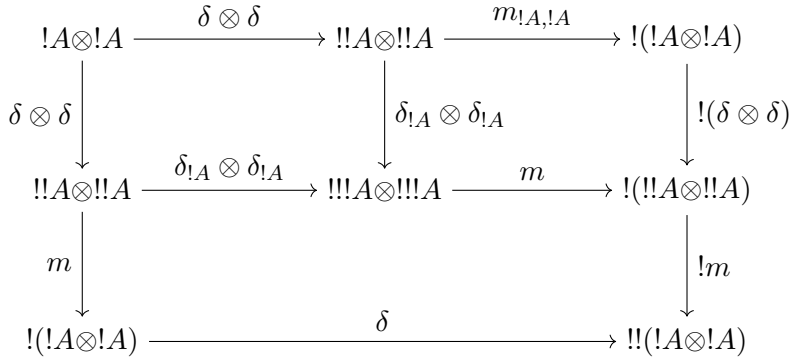


□

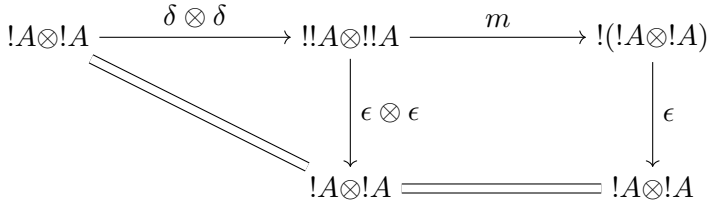
Proposition 3: Given a monoidal comonad $(!, \delta, \epsilon, m_{A,B}, m_1)$ on a symmetric monoidal category $\mathcal{L}$, the pair $(!A \otimes !A, m_{!A,!A} \circ (\delta_A \otimes \delta_A))$ is a !-coalgebra.

Proof: As Bierman notes [2], the proof of coassociativity proceeds by diagram chasing.

In the coassociativity diagram, the upper-left square commutes by the definition of a comonad (specifically, the coassociativity of δ), the upper-right square commutes because m is a natural transformation, and the lower square commutes because δ itself is a monoidal natural transformation.



For counitality, the left-hand triangle holds by the counitality axiom of the comonad, and the remaining square commutes because ϵ is a monoidal natural transformation.



□

4.3 Comonoid Induced by the Comonad

Now we are ready to connect comonad ! to the Weakening and Contraction rules. It turns out, a symmetric monoidal comonad induces a *commutative comonoid structure* on every object of the form $!A$. This comonoid structure is the categorical interpretation of the ability to duplicate and discard resources marked with an exponent.

Given any object A in our category $\mathcal{L}$, we have established in **Proposition 1** that $(!A, \delta_A)$ forms a !-coalgebra. For each such coalgebra, we can construct two distinguished natural transformations:

- $d_A : !A \to !A \otimes !A$ (the *duplication* or *contraction* morphism),
- $e_A : !A \to 1$ (the *discarding* or *weakening* morphism).

These morphisms correspond directly to the Contraction and Weakening rules, as we wanted. Their existence is established through the following derivations:

$$\dfrac{\dfrac{\dfrac{}{!A \vdash !A}\text{Axiom} \quad \dfrac{}{!A \vdash !A}\text{Axiom}}{!A, !A \vdash !A \otimes !A}\text{Right } \otimes}{!A \vdash !A \otimes !A}\text{Contraction} \qquad\qquad \dfrac{\dfrac{}{\vdash 1}\text{Right 1}}{!A \vdash 1}\text{Weakening}$$

To verify that these natural transformations establish a commutative comonoid, we must check several coherence conditions. First, both of them must be *monoidal natural transformations*.

To show that $d_A : !A \to !A \otimes !A$ is a monoidal natural transformation we must show the following diagrams commute for all morphisms $f : A \to B$:

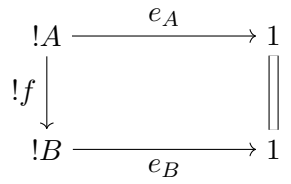
$$\begin{array}{ccc} !A & \xrightarrow{d_A} & !A \otimes !A \\ {\scriptstyle !f}\downarrow & & \downarrow{\scriptstyle !f \otimes !f} \\ !B & \xrightarrow[d_B]{} & !B \otimes !B \end{array}$$

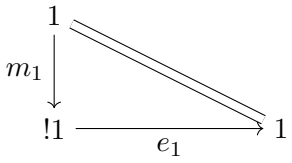
$$\begin{array}{ccc} 1 & \xrightarrow{\lambda^{-1}} & 1 \otimes 1 \\ {\scriptstyle m_1}\downarrow & & \downarrow{\scriptstyle m_1 \otimes m_1} \\ !1 & \xrightarrow[d_1]{} & !1 \otimes !1 \end{array}$$

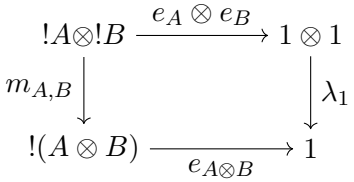
$$\begin{array}{ccccc} !A \otimes !B & \xrightarrow{d_A \otimes d_B} & (!A \otimes !A) \otimes (!B \otimes !B) & \xrightarrow{iso} & (!A \otimes !B) \otimes (!A \otimes !B) \\ {\scriptstyle m_{A,B}}\downarrow & & & & \downarrow{\scriptstyle m_{A,B} \otimes m_{A,B}} \\ !(A \otimes B) & & \xrightarrow{d_{A \otimes B}} & & !(A \otimes B) \otimes !(A \otimes B) \end{array}$$

Here *iso* represents any composition of natural isomorphisms, see page 143 of Bierman's dissertation for an example[2].

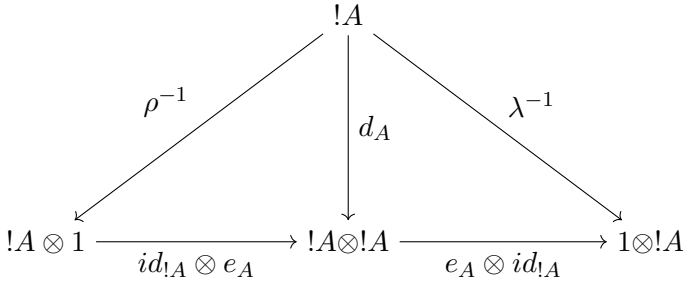
We also show that $e_A : !A \to 1$ is a monoidal natural transformation. Here, for any morphism $f : A \to B$ these diagrams commute:

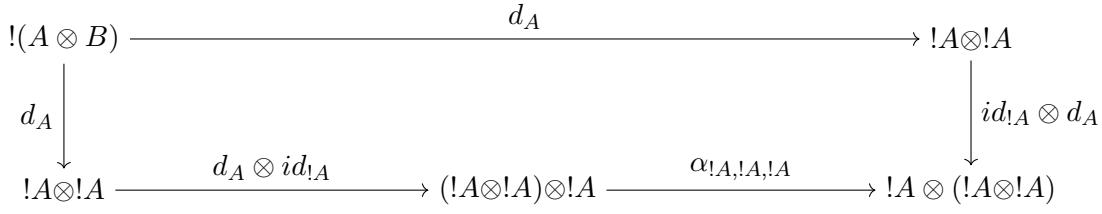


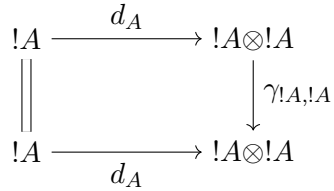




Finally, to establish that $(!A, d_A, e_A)$ forms a *commutative comonoid*, we must verify the comonoid axioms: counitality, and coassociativity, commutativity. These are expressed by the commutativity of the following diagrams:







With this, we have shown that $(!A, d_A, e_A)$ forms a commutative comonoid for every object A.

Now, we can now make explicit the correspondence with the remaining ILL rules. The duplication morphism $d_A : !A \to !A \otimes !A$ clearly corresponds to the *Contraction rule*, which allows a single exponentially-marked assumption $!A$ to be used multiple times. The discarding morphism $e_A : !A \to 1$ corresponds to the *Weakening rule*, which permits an exponentially-marked assumption $!A$ to be ignored entirely. In other words, with this structure any object wrapped in exponential modality ! can escape the confines of structural rules of linear logic.

We can, of course, show that these morphisms exist in our category $\mathcal{L}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{!A \vdash !A}\text{Axiom} \quad \dfrac{}{!A \vdash !A}\text{Axiom}}{!A, !A \vdash !A \otimes !A}\text{Right } \otimes}{!A \vdash !A \otimes !A}\text{Contraction}}{}$$

$$\dfrac{\dfrac{}{\vdash 1}\text{Right 1}}{!A \vdash 1}\text{Weakening}$$

With this analysis complete, we have demonstrated how all four exponential rules correspond to categorical structure on our symmetric monoidal closed category:

- **Promotion** corresponds to the comultiplication $\delta : ! \Rightarrow !!$ of the comonad,
- **Dereliction** corresponds to the counit $\epsilon : ! \Rightarrow \mathrm{Id}$ of the comonad,
- **Contraction** corresponds to the duplication morphism $d_A : !A \to !A \otimes !A$ of the induced comonoid,
- **Weakening** corresponds to the discarding morphism $e_A : !A \to 1$ of the induced comonoid.

Then, every rule has been interpreted. In the next section, we show how these pieces fit together in a definition of a linear category - an established categorical model of ILL.

5 Intuitionistic Linear Logic as a Linear Category

We have now assembled all the necessary categorical components to provide a complete semantics for intuitionistic linear logic with finite products. The categorical structure we have been constructing throughout this chapter is known as a *linear category*, a notion introduced by Bierman [2] as a refinement of Seely's original categorical model [9]. The advantage of Bierman's formulation is that it makes explicit the role of the induced comonoid structure on exponentially-marked objects, which as we have seen corresponds directly to the Weakening and Contraction rules.

Definition 8: Linear Category

A *linear category* consists of the following data:

1. A symmetric monoidal closed category $(\mathcal{L}, \otimes, 1, \multimap)$ with finite products $(\mathcal{L}, \&, \top)$,
2. A symmetric monoidal comonad $(!, \delta, \epsilon, m_{A,B}, m_1)$ on $\mathcal{L}$,

such that:

(i) For every object A, the free !-coalgebra $(!A, \delta_A)$ is equipped with two distinguished monoidal natural transformations:

- $d_A : !A \to !A \otimes !A$ (duplication),

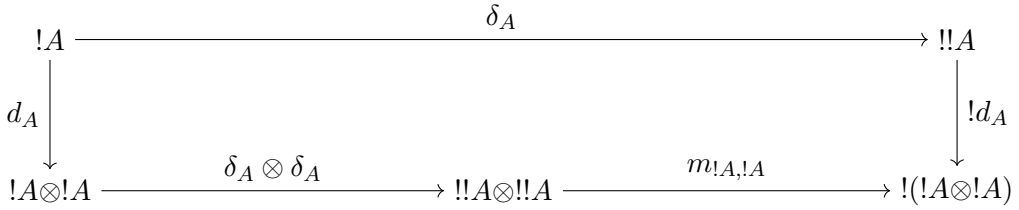
- $e_A : !A \to 1$ (discarding),

which together form a commutative comonoid structure $(!A, d_A, e_A)$, and which are themselves coalgebra morphisms in the appropriate sense.

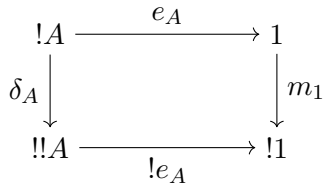
(ii) Whenever $f : (!A, \delta_A) \to (!B, \delta_B)$ is a !-coalgebra morphism, it is also a comonoid morphism from $(!A, d_A, e_A)$ to $(!B, d_B, e_B)$.

We have already verified most of the structure required by this definition in the preceding subsections. The symmetric monoidal closed category with finite products was established in Sections 3.5 and earlier discussions of the additive connectives. The symmetric monoidal comonad structure was constructed in Subsection 4.1, and the induced comonoid structure was derived in Subsection 4.3. What remains is to verify conditions (i) and (ii) explicitly: that d_A and e_A are coalgebra morphisms, and that coalgebra morphisms preserve the comonoid structure.

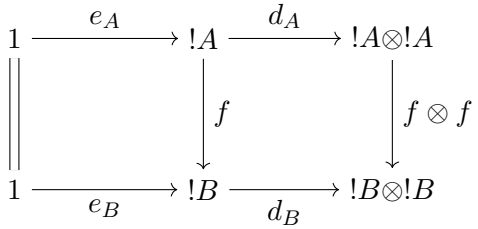
To show that $d_A : !A \to !A \otimes !A$ is a coalgebra morphism from $(!A, \delta_A)$ to the tensor product coalgebra $(!A \otimes !A, m_{!A,!A} \circ (\delta_A \otimes \delta_A))$ (whose existence was established in **Proposition 3**), we must verify that the following diagram commutes:



Similarly, to show that $e_A : !A \to 1$ is a coalgebra morphism from $(!A, \delta_A)$ to the unit coalgebra $(1, m_0)$ (from **Proposition 2**), we must verify that the following diagram commutes:



Finally, we must establish condition (ii): any coalgebra morphism between free coalgebras automatically respects the induced comonoid structure. Specifically, suppose $f : !A \to !B$ is a coalgebra morphism, meaning that $(!f) \circ \delta_A = \delta_B \circ f$. We must show that f is also a comonoid morphism from $(!A, d_A, e_A)$ to $(!B, d_B, e_B)$. This requires verifying two compatibility conditions:



With the verification of conditions (i) and (ii) complete, we have demonstrated that the categorical structure we have been constructing throughout this explainer indeed forms a linear category in the sense of Definition 8 - a sound model of ILL with finite products.

6 Going Further

Bierman's linear category is, of course, not the only categorical model of intuitionistic linear logic, but, likely, the most insightful.

The earliest is the *Lafont category*, which defines a category whose objects are comonoids within a symmetric monoidal closed category $\mathcal{L}$. This approach treats all objects as having comonoid structure, not explicitly distinguishing between linear resources and exponentially-marked resources. It has been shown to be sound.

It was followed by a *Seely category*, which was intended as a more faithful model of linear logic than Lafont. However, Bierman identified soundness issues due to some natural isomorphism conditions in it [2]. The issue was corrected, and the now-sound category was called a *new-Seely category*. Later, Bierman and de Paiva proved that every new-Seely category is a linear category [3], showing Bierman's definition unifies the corrected earlier models.

These models extend to *full classical linear logic* (CLL), which includes classical linear negation $(\cdot)^{\perp}$ and the par connective $\parr$. Modeling CLL requires $*$-autonomous categories or linearly distributive categories [9].

Out of our sources, we encourage further exploration through Bierman's dissertation [2] as it also considers the relationship between ILL and lambda calculus.

7 PL Applications

Linear logic has significantly influenced programming language design through *linear types*. Linear types are *substructural type systems* that, similarly, restrict structural rules to enforce resource management at compile time.

Rust is the most prominent language using linear types. Its ownership and borrowing system implements linear and affine types (affine types allow discarding but not duplication), enforcing memory safety and preventing data races at compile time. Haskell supports linear types through the `linear-base` library and GHC's `LinearTypes` extension. OCaml has experimental support through session types and resource management projects.

Linear types apply to concurrent and distributed systems (via session types) and external resource management (file handles, network connections). The categorical semantics developed in this chapter provides the mathematical foundation for these practical type systems.

8 Conclusion

In this explainer we explored a small but self-contained fragment of linear logic, all the way from sequent calculus to categorical semantics. By interpreting every rule of ILL with categorical constructs, we developed a solid understanding for what our model does every step of the way, and this

makes the abstraction seem almost intuitive. Of course, there is a lot more complexity to address and cases to consider, but we hope that you still were able to appreciate the insights of this field.

9 Exercises

1. Confirm that the Category of Formulas and Proofs is a category by demonstrating that associativity of composition and composition with identity holds

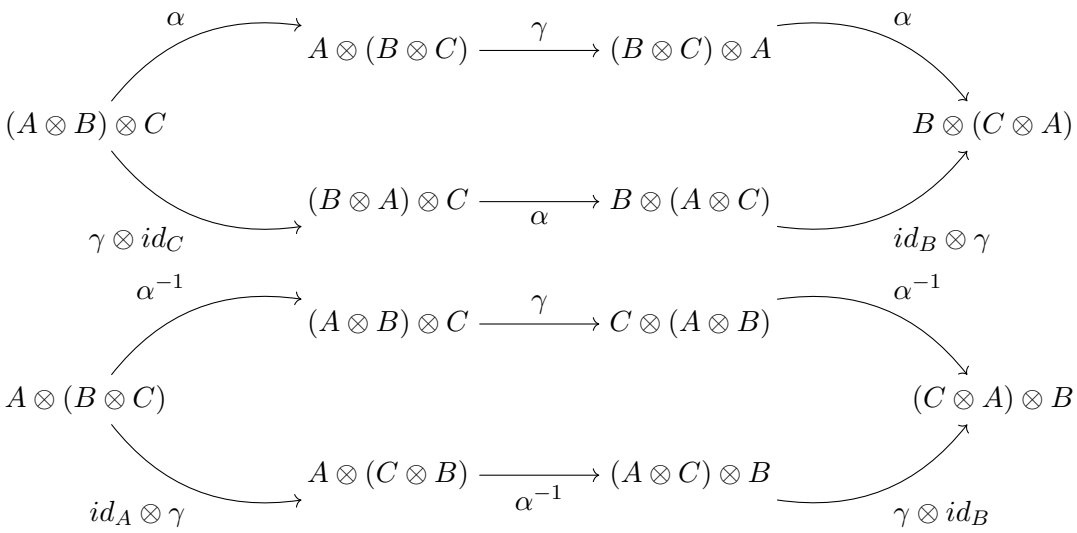
   Verify that $\mathcal{L}$ satisfies the axioms of a category:

   (a) **Left identity:** For any morphism $[\![\pi]\!] : [\![A]\!] \to [\![B]\!]$, show that $[\![\pi]\!] \circ \mathrm{id}_A = \mathrm{id}_B \circ [\![\pi]\!] = [\![\pi]\!]$.

   (b) **Associativity:** For morphisms $[\![\pi_1]\!] : [\![A]\!] \to [\![B]\!]$, $[\![\pi_2]\!] : [\![B]\!] \to [\![C]\!]$, and $[\![\pi_3]\!] : [\![C]\!] \to [\![D]\!]$, show that $([\![\pi_3]\!] \circ [\![\pi_2]\!]) \circ [\![\pi_1]\!] = [\![\pi_3]\!] \circ ([\![\pi_2]\!] \circ [\![\pi_1]\!])$.

2. Show that the $\mathcal{L}$ has finite products. Recall the proof of traditional categorical product and express it as proof equality under cut-elimination.

3. Show that the following diagrams for braiding commute:



4. An alternative definition of a left closed structure is as follows: For every object $A \in \mathrm{Ob}(\mathcal{C})$ we can define two adjoint functors $- \mapsto (A \otimes -) \dashv - \mapsto (A \mapsto B)$. Prove that $\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(B, A \mapsto C)$ are in bijection and explain how that shows that linear implication is an internal hom in our category of formulas and proofs.

5. An alternative definition of the comonad for the exponential modality is as an adjunction between our SMCC $\mathcal{L}$ and a category $\mathcal{M}$ and functors $L \dashv M$ where $L : \mathcal{M} \to \mathcal{L}$ and $M : \mathcal{L} \to \mathcal{M}$. Prove that $\mathcal{M}$ has finite products.

6. Show that the adjunction given in the previous exercise equipped with two natural isomorphisms $m_{A,B} : !A \otimes !B \cong !(A \& B)$ and $m_1 : 1 \cong !\top$ is a symmetric lax monoidal adjunction (in context this is called a "linear-non-linear adjunction").

7. Prove that the symmetric monoidal functor and comonad shown in the previous exercises satisfies the definition of a Seely category.

8. Prove that the forgetful functor U and free functor R are adjoint $U \dashv R$. In other words, prove there is a natural bijection $\phi_{(A,\alpha),X} : \mathrm{Hom}_{\mathcal{C}}(U(A,\alpha), X) \cong \mathrm{Hom}_{\mathcal{C}^!}((A\alpha), (!X, \delta_X))$.